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Finite Differencing and Grid Structure for Mass Conservation about the Pole in Spherical Coordinates

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This note discusses the basic steps used to construct finite difference forms for mass convergence in spherical coordinates for the NMC hurricane model. The original motivation behind this approach was the observation in an axial symmetric prototype of gravity wave generation at the storm center. These modes of horizontal scales around $4\Delta x$, seemed most directly linked to spurious, central pressure changes and lack of exact mass conservation inherent in an early differencing scheme derived from semimomentum differencing concepts.

Since the NMC global model employs a similar semimomentum type differencing and under low damping conditions exhibits $4\Delta x$ waves of apparent polar origin, this note is presented to suggest a possible approach to solution of this numerical problem.

One should take special note in the following of three fundamental characteristics of the approach. First, mass fluxes are defined consistently between adjacent grid boxes. Second, Gauss's theorem expressed in the following form:

$$\int_A D dA = \oint V ds \tag{1}$$

is employed to define the mean mass divergence, $\overline{D}$. In the above equation, V is the normal component of momentum to line S.

By the mean value theorem, we have from Eq. (1)

$$\overline{D} = \frac{\oint V ds}{\int_A dA} \tag{2}$$

Finally, the polar point is not defined as a grid point but rather lies on the boundary of a grid element. This characteristic of the approach is particularly useful if one deals with a symmetric vortex about the polar point but may be of lesser importance in the general case.

A sample of grid elements in the vicinity of the polar point P is illustrated in Fig. 1. The intersection of lines and curves are grid points denoted by integers. Lettered locations represent points where one-dimensional averages are centered. The center of the four-sided grid element is defined by the symbol o and the three-sided elements are defined by center asterisks (*).

Using these definitions, we may express the line integral of the mass flux around the four-sided grid area in Fig. 1 as

$$\oint Vds = -\overline{v}^{\lambda}_{a}\,\Delta\lambda r\,\cos(\phi_o - \frac{\Delta\phi}{2}) + \overline{u}^{\phi}_{b}\,r\Delta\phi + \overline{v}^{\lambda}_{c}\,\Delta\lambda r\,\cos(\phi_o + \frac{\Delta\phi}{2}) - \overline{u}^{\phi}_{d}\,r\Delta\phi \tag{3}$$

Here $\Delta\lambda$ and $\Delta\phi$ are the grid increment in latitude and longitude respectively, r is the mean radius of the earth, and $\overline{u}^{\phi}$ and $\overline{v}^{\lambda}$ are the mean components of momentum normal to the sides of the element.

The area of the four-sided grid element in Fig. 1 is

$$A_o = r^2 \int_{\lambda_a - \frac{\Delta\lambda}{2}}^{\lambda_a + \frac{\Delta\lambda}{2}} d\lambda \int_{\phi_b - \frac{\Delta\phi}{2}}^{\phi_b + \frac{\Delta\phi}{2}} \cos\phi d\phi = 2r^2\,\cos(\phi_o)\Delta\lambda\,\sin\left(\frac{\Delta\phi}{2}\right) \tag{4}$$

Employing (2), we obtain the following expression for the mean divergence over the grid element

$$\overline{D}_o = \frac{\overline{u}^{\phi}_{b} - \overline{u}^{\phi}_{d}}{(r\,\cos\phi_o)\Delta\lambda} \cdot \left(\frac{\frac{\Delta\phi}{2}}{\sin\frac{\Delta\phi}{2}}\right) + \frac{\overline{v}^{\lambda}_{c} - \overline{v}^{\lambda}_{a}}{2r\,\sin\frac{\Delta\phi}{2}} \cdot \cos\frac{\Delta\phi}{2} - \frac{(\overline{v}^{\lambda}_{c} + \overline{v}^{\lambda}_{a})}{2r}\,\tan\phi_o \tag{5}$$

This converges toward the proper definition of divergence in spherical coordinates, since as $\Delta\phi$ and $\Delta\lambda$ approach zero,

$$\sin\frac{\Delta\phi}{2} \to \frac{\Delta\phi}{2} \quad \text{and} \quad \cos\frac{\Delta\phi}{2} \to 1.$$

In a similar manner, we may obtain the line integral of mass flux around the "curved pie wedge" elements centered around the pole. We have

$$\oint Vds = -\overline{v}^{\lambda}_{c}\,\Delta\lambda r\,\cos(\phi_o + \frac{\Delta\phi}{2}) + \overline{u}^{\phi}_{f} r\,\frac{\Delta\phi}{2} - \overline{u}^{\phi}_{g} r\,\frac{\Delta\phi}{2} \tag{6}$$

The wind vector at the pole should be defined in component form to be in the same direction as those east-west components at the southern side of the pie wedge.

This expression when divided by the area of the wedges, i.e.,

$$A_* = r^2 \int_{\lambda_a-\frac{\Delta\lambda}{2}}^{\lambda_a+\frac{\Delta\lambda}{2}} d\lambda \int_{\frac{\Pi}{2}-\frac{\Delta\phi}{2}}^{\frac{\Pi}{2}} \cos\phi d\phi = 2r^2(\Delta\lambda)\ \sin^2\left(\frac{\Delta\phi}{4}\right) \tag{7}$$

yields

$$\overline{D}_* = \frac{(\overline{u}^{\phi}_f - \overline{u}^{\phi}_g)\Delta\phi}{4r\Delta\lambda\ \sin^2\left(\frac{\Delta\phi}{4}\right)} - \frac{\overline{v}^{\lambda}_c\ \cos(\phi_o + \frac{\Delta\phi}{2})}{2r\ \sin^2\left(\frac{\Delta\phi}{4}\right)} \tag{8}$$

This expression, while differing in appearance from Eq. (5), guarantees conservation during mass exchanges with grid elements to the south as well as those adjacent east and west. The area of the wedge elements is not crucial to the conservation property and could be adjusted to satisfy other needs (e.g. to fit requirements in the momentum equations or to adapt the scheme to codes that are more or less written in stone). The only constraint in this regard should be one of approximate size equivalency between adjacent grid elements.

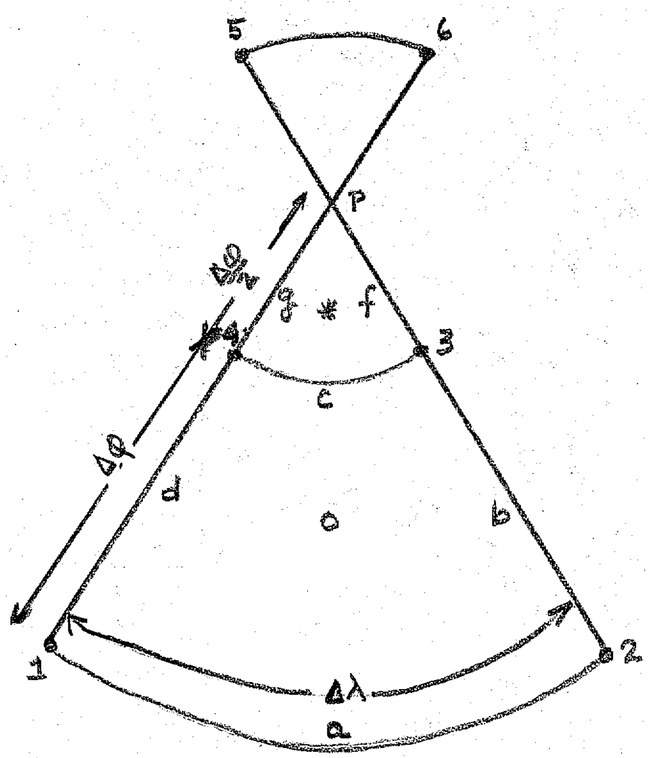

Figure 1.